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2006 J. Phys.: Condens. Matter 18 3031

(http://iopscience.iop.org/0953-8984/18/11/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **18** (2006) 3031–3036 doi:10.1088/0953-8984/18/11/009

Localized oscillatory acoustic pulses

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Received 16 December 2005
Published 27 February 2006
Online at stacks.iop.org/JPhysCM/18/3031

Abstract

A family of three-dimensionally localized sound pulses is studied. These pulses have a plane-wave nature near their core. A subset is characterized by a single dimensionless parameter ka, where k is the wavenumber and a is a length, giving the size of the pulse. The energy and momentum of this subset are evaluated analytically. The energy of the pulse is always greater than the speed of sound times its momentum, in accord with a general theorem for three-dimensionally localized pulses. For large ka this inequality tends to equality, and the energy and momentum are consistent with the pulse being made up of many phonons each of energy $\hbar ck$ and momentum $\hbar k$.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

It has recently been shown that three-dimensionally localized sound pulses always have energy E greater than c times their momentum P [1]. The proof assumes a structureless fluid characterized by an equilibrium density ρ_0 and sound speed c, so it applies to wavelengths large compared to atomic size. This result is in contradistinction to the textbook phonon of wavenumber k, which has energy $\hbar ck$ and momentum $\hbar k$ in the long-wavelength limit. The physical reason for the difference is that the textbook phonon is delocalized (it is represented by a plane wave $e^{i(kz-\omega t)}$); in the plane-wave limit classical hydrodynamics also gives $E = cP$ [1]. The delocalized phonon corresponds well to the experimental situation in inelastic neutron scattering spectroscopy when the neutrons interact weakly with scattering medium, since the interaction is simultaneously with a large volume of the fluid (see for example [2–6]). However, there are experimental situations where the sound pulse localization may have to be taken into account [7–9].

It would be interesting to have experimental data on the energy and momentum of acoustic pulses of various degrees of localization. The example presented in [1] is an extreme case of a non-oscillatory pulse. Here we shall give results for a family of three-dimensionally localized pulses which have a central region over which the space-time dependence is that of a plane wave. The family depends on the wavenumber k and on a characteristic length a. Both the energy and the momentum of the pulse are linear in ka: we find

$$E = \frac{\pi^2}{4}(ka+1)a\rho_0 V_0^2$$
$$cP_z = \frac{\pi^2}{4}(ka)a\rho_0 V_0^2 \tag{1}$$

where V_0 is the magnitude of the velocity potential $V(\mathbf{r}, t)$ at the space-time origin. These formulae follow from the expressions [1]

$$E = \tfrac{1}{2}\rho_0 \int d^3r[(\partial_{ct}V)^2 + (\nabla V)^2]$$
$$c\mathbf{P} = -\rho_0 \int d^3r(\partial_{ct}V)\nabla V \tag{2}$$

which are exact to second order for any solution of the wave equation

$$\nabla^2 V = \partial_{ct}^2 V \tag{3}$$

which satisfies

$$\int d^3r\, \partial_{ct} V = 0. \tag{4}$$

The condition (4) ensures that the pulse energy is purely second order in V; when (4) is not satisfied the energy contains a first-order term proportional to $\int d^3r\, \partial_{ct}V$, which in turn is proportional to the integral over all space of the first-order deviation ρ_1 from the equilibrium density ρ_0, since

$$\rho_1 = -\left(\frac{\rho_0}{c}\right)\partial_{ct}V. \tag{5}$$

Thus (4) ensures mass conservation to first order.

We observe from (1) that $E > cP_z$, in accord with the general result of [1]. Note also that for $ka \gg 1$ the result (1) can be interpreted in terms of phonons of energy $\hbar ck$ and momentum $\hbar k$, the number of phonons forming the pulse being given by

$$N \approx \frac{E}{\hbar ck} \approx \frac{P_z}{\hbar k} = \frac{\pi^2}{4}\frac{a^2\rho_0 V_0^2}{\hbar c}. \tag{6}$$

2. Localized oscillatory solutions of the wave equation

Particle wavepackets satisfying the Schrödinger equation were discovered in the early days of quantum mechanics [10, 11], but localized solutions of the wave equation are relatively recent. One useful family was found by Hillion [12]: define

$$s = \frac{\rho^2}{b + i(z+ct)} - i(z-ct) \tag{7}$$

where $\rho = [x^2+y^2]^{\frac{1}{2}}$ is the distance from the z-axis, and let f be any twice-differentiable function of its argument. Then

$$\psi(\rho, z, t) = \frac{f(s)}{b + i(z+ct)} \tag{8}$$

satisfies the wave equation (3), as may be verified by differentiation. Another route to (8) was given in [13], where we considered the special case $f(s) = e^{-ks}/(s+a)$. In this paper we use the same $f(s)$, set $b = a$, and take the velocity potential of the acoustic pulse to be

$$V(\rho, z, t) = \frac{a^2 V_0 \exp\left[-\frac{k\rho^2}{a+i(z+ct)} + ik(z-ct)\right]}{\rho^2 + [a - i(z-ct)][a + i(z+ct)]}. \tag{9}$$

In the axial region where ρ and $k\rho^2$ are small compared to the length a, and when a is also large compared to $|z \pm ct|$, the velocity potential has the plane-wave form: $V \to V_0 \exp[\mathrm{i}k(z - ct)]$.

We first check whether (4) is satisfied, i.e. whether

$$\int \mathrm{d}^3 r \, \partial_{ct} V = 2\pi \partial_{ct} \int_{-\infty}^{\infty} \mathrm{d}z \int_0^{\infty} \mathrm{d}\rho \, \rho V = 0. \tag{10}$$

In performing the integration over the lateral coordinate ρ we keep z constant, and change to s as variable:

$$2 \int_0^{\infty} \mathrm{d}\rho \, \rho \, V = a^2 V_0 \int_{-\mathrm{i}(z-ct)}^{\infty} \mathrm{d}s \, \mathrm{e}^{-ks}/(s + a). \tag{11}$$

The right-hand side can be evaluated as an exponential integral, but we just need the fact that the result is a function of ka and $k(z-ct)$. The integration over z now gives a function independent of time (we change to $z - ct$ as variable of integration), so (10) is zero, as required.

3. Pulse pressure, density change, and velocity components

The first-order density change ρ_1 due to the pulse is given by (5), and the pulse pressure is $p_1 = c^2 \rho_1$ [14], so both ρ_1 and p_1 are given by the time derivative of the velocity potential. The fluid velocity is the gradient of the velocity potential, so it has the transverse and longitudinal components

$$v_\rho = \partial_\rho V, \qquad v_z = \partial_z V. \tag{12}$$

Thus all physical quantities of the pulse are given by the three derivatives $\partial_{ct} V$, $\partial_\rho V$ and $\partial_z V$.

The velocity potential given in (9) is complex, $V = V^r + \mathrm{i}V^i$ where V^r and V^i are real functions of ρ, z and ct. Both V^r and V^i are solutions of the wave equation, and both satisfy the condition (4), so the function (9) gives us two pulses.

In the evaluation of the energy and momentum integrals (2) we can set $t = 0$, since it has been verified that the total energy and total momentum of the pulse are independent of time [1]. (This must be so, since scattering and viscous dissipation have been neglected.) Thus, for the energy and momentum integrals, we need only the $t = 0$ values of $\partial_{ct} V$, $\partial_\rho V$ and $\partial_z V$. These are, for the real and imaginary parts of V,

$$\begin{aligned} d(k\alpha^2 V_0)^{-1} \, \partial_{ct} V^r &= 2\alpha\zeta\eta(\eta - 1)\cos\eta\zeta + \left[2\alpha(\alpha\eta + 1) + \eta^2(\zeta^2 - \alpha^2)\right]\sin\eta\zeta \\ d(k\alpha^2 V_0)^{-1} \, \partial_{ct} V^i &= 2\alpha\zeta\eta(\eta - 1)\sin\eta\zeta - \left[2\alpha(\alpha\eta + 1) + \eta^2(\zeta^2 - \alpha^2)\right]\cos\eta\zeta \end{aligned} \tag{13}$$

$$\begin{aligned} d(k\alpha^2 V_0)^{-1} \, \partial_\rho V^r &= -2k\rho\left[(\alpha\eta + 1)\cos\eta\zeta + \eta\zeta\sin\eta\zeta\right] \\ d(k\alpha^2 V_0)^{-1} \, \partial_\rho V^i &= -2k\rho\left[(\alpha\eta + 1)\sin\eta\zeta - \eta\zeta\cos\eta\zeta\right] \end{aligned} \tag{14}$$

$$\begin{aligned} d(k\alpha^2 V_0)^{-1} \, \partial_z V^r &= 2\zeta\left[\alpha\eta(\eta - 1) - 1\right]\cos\eta\zeta - \eta\left[\eta\alpha^2 + \zeta^2(2 - \eta)\right]\sin\eta\zeta \\ d(k\alpha^2 V_0)^{-1} \, \partial_z V^i &= 2\zeta\left[\alpha\eta(\eta - 1) - 1\right]\sin\eta\zeta + \eta\left[\eta\alpha^2 + \zeta^2(2 - \eta)\right]\cos\eta\zeta. \end{aligned} \tag{15}$$

We use the dimensionless variables

$$\alpha = ka, \qquad \zeta = kz, \qquad \eta = 1 + \frac{\rho^2}{a^2 + z^2}. \tag{16}$$

The common denominator d is given by

$$d = \eta^2(\alpha^2 + \zeta^2)^2 \mathrm{e}^{\alpha(\eta - 1)}. \tag{17}$$

In performing the spatial integrals in (2) to evaluate E and P_z, we first keep z constant and integrate with respect to ρ, using

$$\int_0^{\infty} \mathrm{d}\rho \, \rho = \tfrac{1}{2}(a^2 + z^2) \int_1^{\infty} \mathrm{d}\eta = \tfrac{1}{2}k^{-2}(\alpha^2 + \zeta^2) \int_1^{\infty} \mathrm{d}\eta. \tag{18}$$

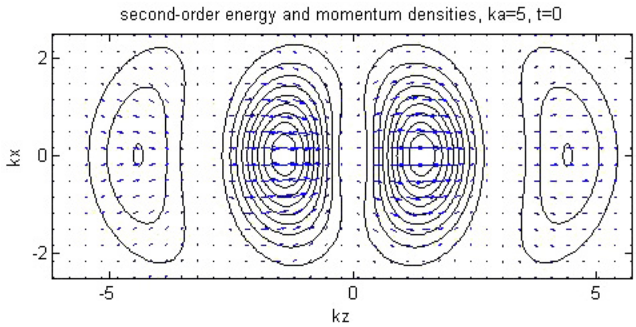

Figure 1. Second-order energy and momentum densities for $ka = 5$ at $t = 0$, derived from the real part of the velocity potential (9). The contours are of the energy density, the arrows give the direction and magnitude of the momentum density. The imaginary part of the velocity potential gives similar densities, but with a maximum centred on the origin (when $t = 0$).

Then the z or ζ integrations are performed. The intermediate steps contain the exponential integral Ei, but the final results, given in (1), are simple. (The real and imaginary parts of the complex velocity potential (9) give the same energy and momentum values.)

4. Energy and momentum densities

The on-axis ($\rho = 0$) values of the respective velocity potential derivatives at $t = 0$ are $\partial_\rho V^{r,i} = 0$ and

$$\begin{aligned} d(k\alpha^2 V_0)^{-1}\,\partial_{ct} V^r &= \frac{\sin\zeta}{\alpha^2+\zeta^2} + \frac{2\alpha\sin\zeta}{(\alpha^2+\zeta^2)^2} \\ d(k\alpha^2 V_0)^{-1}\,\partial_{ct} V^i &= \frac{-\cos\zeta}{\alpha^2+\zeta^2} - \frac{2\alpha\cos\zeta}{(\alpha^2+\zeta^2)^2} \end{aligned} \tag{19}$$

$$\begin{aligned} d(k\alpha^2 V_0)^{-1}\,\partial_z V^r &= \frac{-\sin\zeta}{\alpha^2+\zeta^2} - \frac{2\zeta\cos\zeta}{(\alpha^2+\zeta^2)^2} \\ d(k\alpha^2 V_0)^{-1}\,\partial_z V^i &= \frac{\cos\zeta}{\alpha^2+\zeta^2} - \frac{2\zeta\sin\zeta}{(\alpha^2+\zeta^2)^2}. \end{aligned} \tag{20}$$

Thus the second-order energy and momentum densities near the origin are proportional to $\sin^2 kz$ and $\cos^2 kz$ respectively for the real and imaginary parts of the velocity potential, since they are given by

$$e(\mathbf{r},t) = \tfrac{1}{2}\rho_0\left[(\partial_{ct}V)^2 + (\partial_\rho V)^2 + (\partial_z V)^2\right] \tag{21}$$

$$p_z(\mathbf{r},t) = -\rho_0(\partial_{ct}V)(\partial_z V). \tag{22}$$

Figure 1 shows the energy and momentum second-order densities for the real part of the velocity potential. (The first-order parts integrate to zero, as discussed above, and in [1].) The pulse is shown for the intermediate value $ka = 5$. For large ka there are many oscillations within the pulse, and small convergence and divergence. For small ka there are few oscillations, and a large degree of convergence toward the focal region (at the origin) for negative times, followed by the same degree of divergence from the focal region at positive times. At $t = 0$ the pulse is centred on the focal region. As $k \to 0$ we regain the total energy and momentum values given in equation (35) of [1], when $a = b$ (in this case the velocity potential tends to the one used there).

Figure 2 shows the time development of the same pulse in snapshots of the second-order energy density at three successive times. We see again the convergence into the focal region, followed by spreading as the pulse propagates out from the focal region.

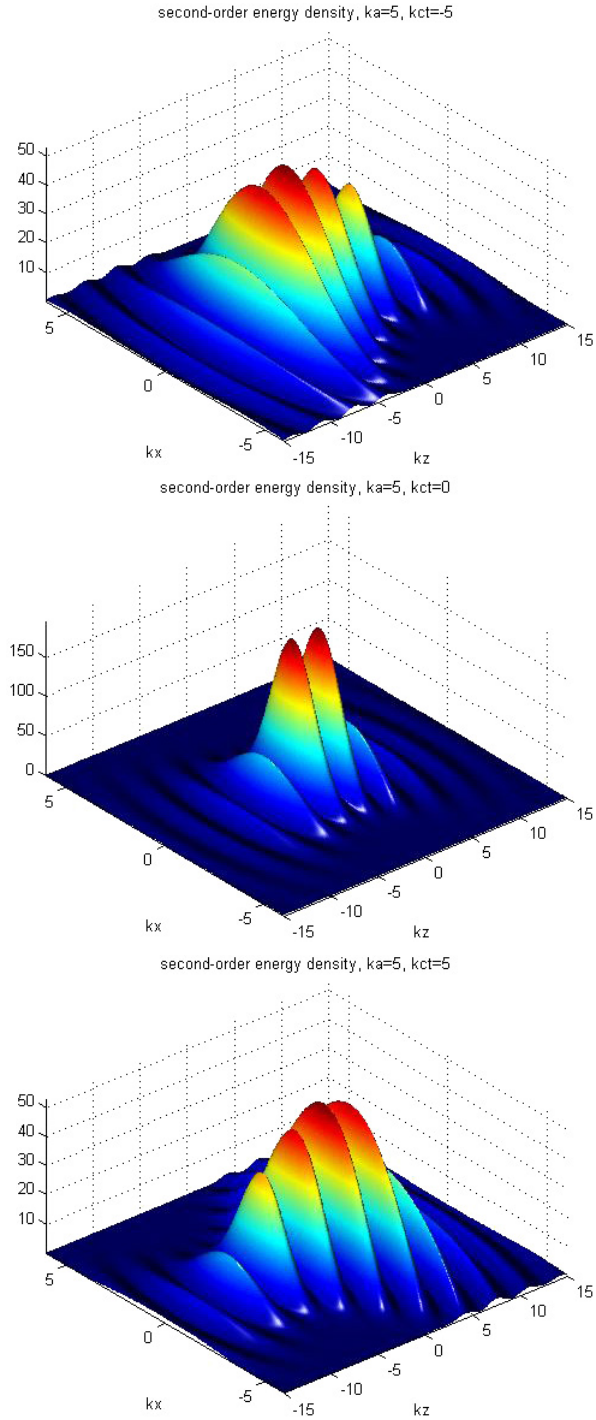

Figure 2. The second-order energy density of the $ka = 5$ pulse of figure 1, at consecutive times $kct = -5, 0, 5$. Note the convergence into the focal region for $t < 0$, the maximum compression of the pulse at $t = 0$, and the pulse spreading for $t > 0$. (The transverse scale is expanded for clarity; this makes the wave crests seem sharper than they are.)

5. Discussion

We have presented analytic results for the energy and momentum densities of sound pulses derived from a family of solutions of the wave equation. In contradistinction to the textbook long-wavelength phonon case, the energy is always greater than c times the momentum. However, in the limit of large ka, we regain approximate equality between E and cP. This exactly parallels the results for localized electromagnetic pulses [15, 16, 13], for which the energy is always greater than the speed of light times the momentum, in contradistinction to Einstein's photon [17]. The reason is the same in both cases: three-dimensionally localized solutions of the wave equation necessarily converge or spread as they propagate.

There appear to be no experimental data on the energy and momentum of localized acoustic pulses. Even in the absence of such data, analysis of some recent experiments [7–9] using localized sound pulses would be of interest.

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